Spanning traceroutes over modular networks and general scaling degree distributions

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We analyze the class of networks characterized by modular structure where a sequence of ℓ Erdös-Renyi random networks of size $N \gg \ell$ with random average degrees is joined by links whose structure must remain immaterial. We find that traceroutes spanning the entire macronetwork exhibit scaling degree distributions $P(k) \sim k^{-\gamma}$, where γ depends on how the degrees of the joined clusters are distributed. We thus suggest that yet another mechanism for the dynamic origin of arbitrary power-law degree distributions observed in natural and artificial networks, many of which belong to the range $2 \leq \gamma \leq 3$, may be found in random processes on modular networks.

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I. INTRODUCTION

Modularity is a central organizing principle of complex networks, where the structure of the subdivision into coherent groups over multiple scales reflects functional units reminiscent of ecological niches in ecology, modules in biochemical networks, or communities of social networks [1–4]. Relevant to our purposes, it was shown [5] by exploring a possible bias in probing the scale-free features of very large networks that the degree distribution $P(k)$ of a tree spanning an Erdös-Rényi (ER) random graph follows a power law $P(k) \sim k^{-\gamma}$ with $\gamma = 1$ regardless of its average degree. Given the wide range of γ values observed in natural and artificial networks [2,6,7,20], the extent of possible implications of the result thus seemed somewhat limited. However, subsequent work [8–10] showed that estimates of the scaling exponents of the degree distribution of scale-free networks sampled by extracting spanning trees (the so-called traceroute probes) are generally smaller than the actual exponents. Note that this is true when the probes are more than 1 and the targets are not the whole set of nodes of the network because higher degree nodes are more likely to be mapped correctly than low degree ones. Furthermore, traceroute mapping of networks is obviously more reliable when the underlying network is rather sparse [10]. Thus the centrality of the sampling of networks by static or dynamic approaches is evident [7,9]. Petermann and De Los Rios [9], in particular, showed that different generation algorithms may lead to different topological properties, an idea that has been exploited to show analytically that the observed power-law degree distributions can rather be an artifact of biases affecting the sampling techniques.

Overall, studies on modularity of large networks [1], on clustering and community structure in ecological and social systems aiming at extracting their hierarchical organization [11–17], or on limit scaling properties of spanning trees of arbitrarily connected nodes [18] offer broad insight into several relevant network phenomena.

Moreover, it was shown that dynamical processes taking place on networks may sample specific evolving subnetworks whose topology may not necessarily be the same of the underlying domain [10,19]. The emerging degree distributions, e.g., studied by mean-field arguments both for single-source and multiple-source cases and applied to the specific example of the traceroute exploration of networks, have been shown to provide a qualitative improvement in the understanding of dynamical sampling and of the interplay between dynamics and topology in large networks such as the internet [19].

In this paper we derive the degree distribution of spanning traceroutes extracted from assembled heterogeneous random graphs organized in a modular structure, aiming in particular at probing whether the aggregates exhibit topologies endowed with the scale-free characters of the type observed in natural and artificial networks [2,6,20]. Because we claim that patchworking of chance-dominated locally homogeneous aggregate clusters is indeed a reasonable candidate mechanism for the generation of very large networks [1], its overall properties are deemed relevant. In particular, if the most relevant processes that take place on the network concentrate around a treelike backbone such as a spanning tree or a traceroute (that is, on a very sparse subset preserving the connectivity and some relevant features of the global network), our results suggest how general scale-free characters of key topological features such as the degree distribution may naturally emerge.

II. SPANNING TRACEROUTES AND PATCHWORKING OF RANDOM GRAPHS

We start from the classical random graph model by ER. Starting with N nodes each of the possible $N(N-1)/2$ bonds

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is an edge of the graph with probability q, and the average degree is $\langle k\rangle = q(N-1)$ [21]. We will refer to the ER random graph model with N nodes and probability q by $G(N,q)$. The number of links will be denoted by L. The generation of a breadth-first search (BFS) spanning tree starts from an assigned node (the root) and adds to the tree all the available links that point toward yet unexplored nodes. It is clear that each spanning tree will have approximately degree $k(1) \simeq \hat{k}(1) = d = q(N-1)[\simeq 1+q(N-1)(1-q)^0]$ at the root because all the nodes of the graph are unexplored. Once the second node is processed, a neighbor of the root, it may potentially be connected to each one of the remaining $N-1-d$ nodes with the same probability q; thus its degree will be approximately $\hat{k}(2) = 1+q(N-1-d) = 1+q(N-1)(1-q)^1$. At the ith step, the degree will be estimated by $\hat{k}(i) = 1+qr(i)$, where $r(i)$ is the approximate number of nodes not yet reached by the tree, given inductively by $r(i) = r(i-1)-[\hat{k}(i-1)-1]$. Thus, for $i \geq 2$, it follows that

$$\begin{aligned}\hat{k}(i) &= 1 + q(r(i-1) - \hat{k}(i-1) + 1)\\ &= 1 + qr(1)(1-q)^{(i-1)} = 1 + q(N-1)(1-q)^{(i-1)}.\end{aligned}$$

It is clear that the sequence $\{\hat{k}(i)\}$ is strictly decreasing from $\hat{k}_{max} = \hat{k}(1) = q(N-1)$ to $\hat{k}_{min} = \hat{k}(N) = 1+q(N-1)(1-q)^{N-1}$ [24].

It follows that the cumulative histogram of the degrees ($H^{>}(k) \coloneqq$ No.{nodes with degree $\geq k$}) is approximated by the sequence $\{(\hat{k}(i),i) \mid i=1,\ldots,N\}$. Indeed, because $k(1)$ is the degree of the root of the BFS tree, $H^{>}(k) \equiv 0$ for $k > k(1)$ and $H^{>}(k(1)) = 1$. Because the sequence $\{\hat{k}(i)\}$ is strictly decreasing, there should be two nodes with degree $\geq k(2)$ and three nodes with degree $\geq k(3)$ such that $H^{>}(k(i)) \equiv i$. By inverting the formula for $\hat{k}(i)$ we obtain $H^{>}(k) \simeq \hat{i}(k) = N + (1/\ln(1-q))\ln((k-1)/(\hat{k}_{min}-1))$ if $\hat{k}_{min} \leq k \leq \hat{k}_{max}$, $H^{>}(k) = N$ if $k < \hat{k}_{min}$, and $H^{>}(k) = 0$ if $k > \hat{k}_{max}$. The degree distribution $P_q(k)$ is estimated as minus the derivative with respect to k of $\hat{i}(k)/N$, which is zero outside the range between the lower and upper cutoffs, $\hat{k}_{min}$ and $\hat{k}_{max}$, whereas in the continuous limit one has

$$P_q(k) \simeq \frac{1}{N|\ln(1-q)|}\frac{1}{k-1}, \quad \hat{k}_{min} \leq k \leq \hat{k}_{max}, \tag{1}$$

which is a power law with exponent of -1. This proof complements previous ones [5]. The good agreement with numerical simulations can be observed in Fig. 1.

Patchworking different ER graphs is now analyzed. Specifically, we generate a modular network by picking a sequence $q_1,\ldots,q_\ell$ from a certain probability distribution with density $w(q)$ in the interval (0,1) and fixing a number of nodes $N \gg \ell$. We generate a sequence of ER graphs $G_1,\ldots,G_\ell$ such that $G_i \in G(N,q_i)$ and refer to these graphs as the patches or the local communities.

The choice of a fixed graph (community) size N is not restrictive as it can be shown that a sequence of variably

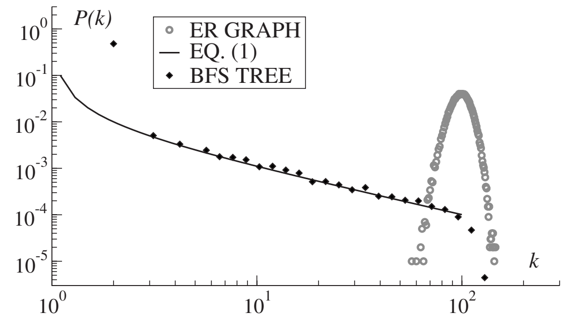

FIG. 1. Sampled degree distribution of a BFS tree (shown here using logarithmic binning) in an Erdös-Rényi random graph $G(10^5,0.001)$ compared with the analytical prediction [Eq. (1)]. The sampled degree distribution of $G(10^5,0.001)$ is also shown.

sized communities can be substituted without affecting the subsequent results by a constant size sequence through a suitably tuned probability distribution density $w(q)$. This stems from the limit of large N for Eq. (1) which depends only on the average cluster degree $\langle k\rangle = q(N-1)$.

The patchwork $G_{N,\ell}^{w(q)}$ is thus obtained by adding a small number of new random links among nodes belonging to different patches until the resulting network becomes connected. Those new links act as gateways among distinct communities (Fig. 2). As long as $N \gg \ell$, the number of gateways proves irrelevant. Compared to edges and connections inside the patches, the gateways are few but crucial to maintain global connectivity, as shown for instance by small-world networks whose diameter may decrease significantly through minor rewiring without affecting the clustering of the whole aggregate [23].

III. RESULTS AND DISCUSSION

Trees spanning the whole network propagate inside single subgraphs as if they were isolated because of the small number of gateway links. When the spanning process passes through one of the gateways, the tree generation algorithm restarts on the next community according to the same rules valid for individual ER graphs. Because the number of gateways is small, we claim that the histogram of the degrees of the global spanning tree may be well approximated by the sum of the histograms of the single spanning trees built over the isolated communities. As a result, the overall degree probability distribution $P(k)$ is obtained by averaging the degree distributions of the isolated communities. This is confirmed by numerical simulations (see Fig. 4) and holds also in the limit of $N \to \infty$, $\ell \to \infty$ while $N/\ell \gg 1$. In such a case, we can estimate exactly the degree distribution of spanning trees over a patchwork of several ER graphs $G(N,q_i)$, with $q_i \sim w(q)$, as follows:

$$P(k) = \frac{1}{\ell}\sum_i P_{q_i}(k) \underset{\ell\to\infty}{\to} \int_0^1 P_q(k)w(q)dq. \tag{2}$$

From Eqs. (1) and (2) it follows that the behavior of P is dominated by the small q behavior of $w(q)$.

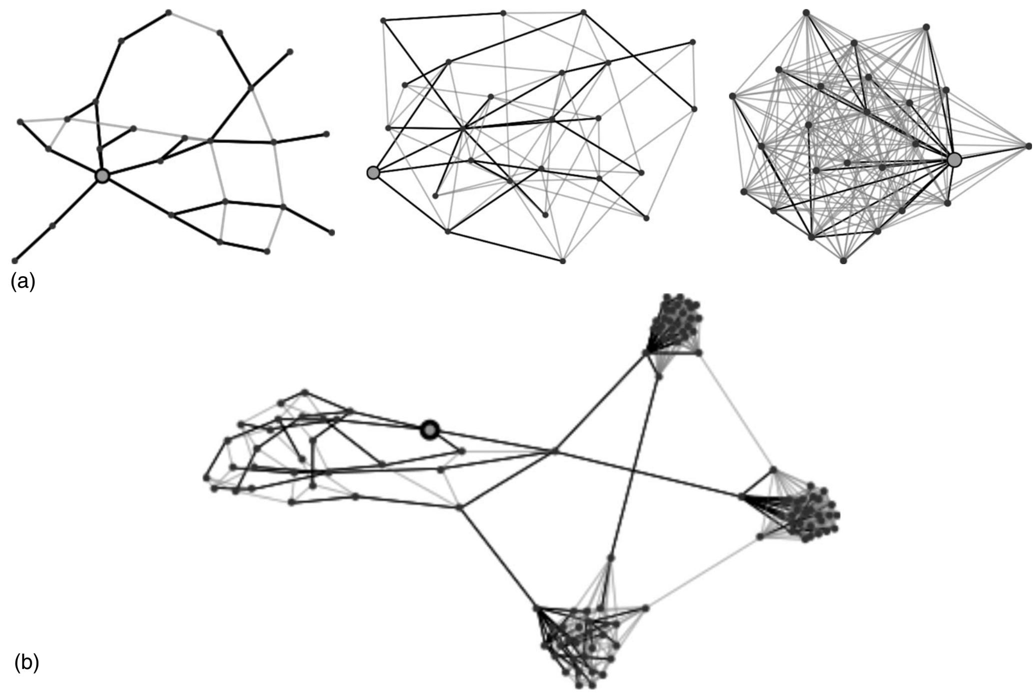

FIG. 2. (a) BFS spanning trees in Poisson random graphs with $N=25$ nodes and $L=35,70,200$ links. The large dot is the root of the tree; (b) modular network obtained by connecting four ER networks with $N=30$, $L=60,120,180,240$ $[L=qN(N-1)/2]$.

We now analyze a few relevant cases. We first assume that

$$w(q)=\begin{cases} C_\alpha q^{-\alpha}[1+h(q)] & \text{for } 0<q_{min}\le q\le q_{max}\le 1 \\ 0 & \text{otherwise,}\end{cases}$$

where $h(q)\xrightarrow{q\to 0} 0$ and $\alpha>0$. If $\alpha\le 1$ the lower cutoff, q_{min}, may also be zero. Otherwise, owing to normalization, the lower cutoff is strictly positive. The integral is dominated by the small q behavior of the integrand and in the range $k_L \equiv q_{min}(N-1)<k<q_{max}(N-1)\equiv k_U$. Equation (2) leads to the following finite-size scaling form [22]:

$$P(k)\sim\int_{k/(N-1)}^{1}\frac{1}{N}\frac{1}{\ln\frac{1}{1-q}}\frac{1}{k-1}w(q)dq$$

$$\sim\frac{1}{N}\frac{C_\alpha}{k-1}\int_{k/(N-1)}^{q_{max}}(q^{-\alpha-1})dq\propto k^{-\alpha-1}f\left(\frac{k}{k_U}\right),\qquad(3)$$

where the scaling function $f(x)\propto(1-x^{\alpha})$ when $0<x<1$ and zero otherwise. Notice that when $x=k/k_U\ll 1$, $f(x)\sim\text{const}$, a power-law behavior is recovered (see Fig. 3). On the other hand in the range $k<k_L$, Eq. (2) gives

$$P(k)\sim\frac{1}{k-1}\int_{q_{min}}^{q_{max}}\frac{w(q)}{\ln\frac{1}{1-q}}dq\sim\tilde{C}k^{-1},\qquad(4)$$

where the k^{-1} behavior of ER networks is recovered for k smaller than the smallest average degree of the communities. Extensive numerical investigation confirms the two separate regimes and the finite-size effect. In Fig. 4 an example of patchwork network with $\alpha=1.0$ is reported.

The case $\alpha=0$ must be treated separately. It corresponds to the case when $w(q)$ approaches a constant in the small q limit which has the same scaling behavior as the case of a uniform $w(q)\equiv 1/(q_{max}-q_{min})$. For $k<q_{min}(N-1)$ the k^{-1} trend is unchanged, while for $k_L<k<k_U$ the degree distribution is estimated by

$$P(k)\sim\frac{C_0}{Nk}\int_{k/(N-1)}^{q_{max}}\frac{1}{\ln\frac{1}{1-q}}dq=k^{-1}f_0\left(\frac{k}{k_U}\right)\qquad(5)$$

with the scaling function $f_0(x)\propto -\ln x$. Differently with the previous case where $f(x)\to\text{const}$ in the small x limit, there exist logarithmic corrections to the pure power-law behavior. These corrections are generally expected when the power-

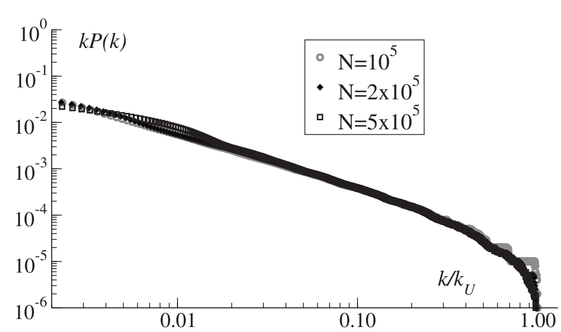

FIG. 3. Scaling function $\tilde{f}(k/k_U)=kP(k)$ [Eq. (5)] from numerical experiments on patchwork networks composed of ten communities with $N=10^5$, 2×10^5, and 5×10^5 nodes and $w(q)\sim q^{-1}$ in the range $q_{min}=10/N$ and $q_{max}=3000/N$. Curves corresponding to different q_{max} (and thus k_U) indeed collapse around a function of k/k_U with $q_{max}(N-1)\equiv k_U$.

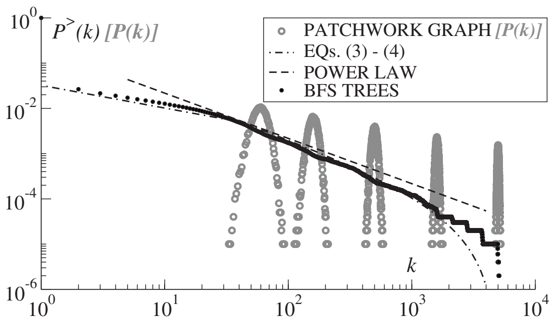

FIG. 4. Sampled cumulative degree distribution $P^>(k)$ of a single BFS tree extracted from a patchwork network composed of five clusters with $N=20\,000$ nodes and $\langle k\rangle =50,158,500,1581,5000$, i.e., sampled according to $w(q)\propto q^{-\alpha}$ with $\alpha=1.0$. This distribution is indistinguishable from the average distribution corresponding to BFS trees extracted from the separate communities. Also shown: first-order approximation (3) for $P^>(k)$, incorporating the finite-size effect and the k^{-1} regime for $k<q_{min}(N-1)$ as in Eq. (4), and the power-law prediction $P^>(k)\propto k^{-1}$ [i.e., $P(k)\propto k^{-2}$]. The gray symbols represent the degree distribution density $P(k)$ of the patchwork network.

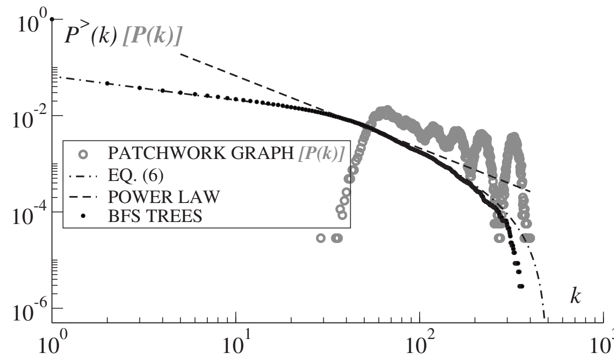

FIG. 6. Sampled cumulative degree distribution $P^>(k)$ of ten BFS trees extracted from a patchwork network composed of seven clusters with $N=5000$ nodes and $\langle k\rangle =60,70,90,120,160,220,325$, i.e., sampled according to $w(q)\propto q^{-1.5}(1-q)^{-2.5}$. Also shown: the analytical prediction for $P^>(k)$ according to Eq. (6) and the power law $P^>(k)\propto k^{-1.5}$ [i.e., $P(k)\propto k^{-2.5}$]. The gray symbols represent the degree distribution density $P(k)$ of the patchwork network.

law exponent becomes 1 since it is the limiting case where the power-law behavior cannot be extended at arbitrary large k due to normalization of $P(k)$ (Fig. 5). The $\alpha<0$ case leads to the same scaling behavior as the $\alpha=0$ case.

The logic behind the choice made for $w(q)$ is simply to provide an ansatz on how $w(q)$ behaves at small q. It may simply go to a constant, implying $\alpha\leq 0$, leading to the universal behavior $P(k)\sim k^{-1}$, or it could diverge. Thus the former case represents a generalization of the nice work done by Clauset and Moore [5]. In the latter case we have assumed a power-law divergence at small q. A more general or realistic case can be well approximated by an appropriate linear combination of power laws. Since $P(k)$ is linear in w [see Eq. (2)] this would lead to $P(k)$ itself being a linear combination of power laws for which we have an exact analytical expression.

To further test the analytical predictions with numerical simulation we have considered the case $w(q)=C_{\alpha\beta}q^{-\alpha}\times(1-q)^{-\beta}$ in $[q_{min},q_{max}]$ and 0 outside. For $k\geq q_{min}(N-1)$, the degree distribution is given by the exact expression in terms of the incomplete Euler function B,

$$P(k)\sim\frac{1}{N}\frac{C_{\alpha\beta}}{k-1}\left[B(q_{max};-\alpha,-\beta+1)-B\left(\frac{k}{N-1};-\alpha,-\beta+1\right)\right],\qquad(6)$$

which has the same scaling behavior as in Eqs. (3) and (4). Figure 6 compares relevant numerical results.

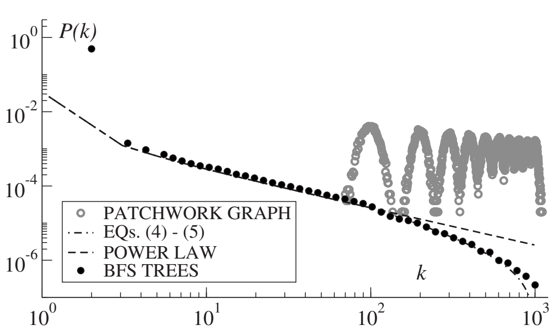

FIG. 5. Sampled degree distribution $P(k)$ of 100 BFS trees (shown using logarithmic binning) extracted from a patchwork network composed of five clusters with $N=20\,000$ nodes and $\langle k\rangle =50,158,500,1581,5000$, i.e., sampled with $w(q)\propto q^{-\alpha}$ with $\alpha=0$, and first-order analytical predictions for $P(k)$ [Eqs. (4) and (5)]. The gray symbols represent the $P(k)$ of the patchwork network.

Note that the proposed merging of heterogeneous ER graphs recalls the study of the interplay between topological and dynamical properties of large networks [19], which is currently of primary interest in the study of complex systems. While therein general methods have been proposed to investigate the topological properties of growing clusters dynamically defined by spreading processes, in this paper we predefine the topologies that are sampled in a sense addressing the ontogeny of local networks rather than the sampling process itself.

Dynamic processes start from a single source to span the whole network, reaching all nodes only once allowing to compute analytically or numerically the degree distribution of the emerging treelike structures because during its evolution the dynamics samples the local structure of the underlying network. As the sampling rate depends on the dynamical properties, the degree distribution of the emerging subnetwork may differ considerably from that of the original network, including generalizations to study sampling induced by multiple-source processes. The degree distribution $\tilde{P}(k)$ of a subnetwork is related to the degree distribution $P(k)$ of the underlying one by the relation [19]

$$\tilde{P}(k)=\sum_{\ell=k}^{\infty}P(\ell)Q(k|\ell),\qquad(7)$$

where $P(\ell)$ is the degree distribution that defines the probability of picking up a node of degree ℓ in the original network and $Q(k|\ell)$ is the conditional probability of observing a node of degree k in the subnetwork if its real degree in the complete network is ℓ. The parallel with Eq. (2) is evident. However, within a dynamical framework the sampling probability would depend on the temporal evolution of the process as initially the neighborhood of interfacial nodes is mainly composed of unreached nodes, whereas in the final stage of the dynamics most of the nodes have already been visited. The relevant probabilities are thus replaced by time-dependent quantities that are defined by the evolution rule of the dynamical processes itself. In our formalism this is substituted by static sampling on the local clusters born independently and randomly assembled, provided the specification of the connectivity that provides the overall aggregate nature remains immaterial.

The present formalism can be extended to study (e.g., numerically) the effect of degree-degree correlations or quenched disorder that has not been considered in this paper.

IV. CONCLUSIONS

We have proposed a mechanism for the dynamic origin and the structure of very large graphs, possibly heterogeneous, probed by traceroutes. We expect that our analysis will allow a better understanding of the ontogeny of large networks, as well as on the functional interplay between a network and the dynamical processes evolving on it.

Implications are foreseen for the design of a new generation of efficient probing algorithms, say for internet applications, and toward building accurate network models. It must be emphasized that in the case studied here, the sampled network is a tree. However, networks such as internet are not trees; on the contrary, the mapping projects yield maps with large clustering coefficients. Our study is limited to spanning trees, however, which we believe represent an important case. The many-point correlation functions in any context, not only in the network field, are important and should be studied although this is not the focus of the present contribution.

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[24] Note that, for finite values of N, the noninteger approximation of the degree of the leaves of the tree, $\hat{k}_{min} \equiv \hat{k}(N)$, approaches 1 from above in the large N limit. Thus in the degree distribution $P_q(k)$ given by Eq. (1) there is no divergence at $k=1$ because $k \geq \hat{k}_{min} > 1$.